Vektorski identiteti ($\triangledown$), Gauss, Stokes, Maxwell

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21. listopada 2009.

Skalarni i vektorski produkt vektora u pravokutnom koordinatnom sustavu

Neka su $\mathbf{i}$, $\mathbf{j}$ i $\mathbf{k}$ jedinični vektori duž x, y i z osi pravokutnog koordinatnog sustava.

Skalarni produkt vektora: $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ i $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$:

$$\mathbf{a} \cdot \mathbf{b} \equiv a_x b_x + a_y b_y + a_z b_z \tag{1}$$

Vektorski produkt vektora $\mathbf{a}$ i $\mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} \equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \mathbf{i}\,(a_y b_z - a_z b_y) + \mathbf{j}\,(a_z b_x - a_x b_z) + \mathbf{k}\,(a_x b_y - a_y b_x) \tag{2}$$

Vektorski identiteti (1)

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \tag{3}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \tag{4}$$

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \\ &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \end{aligned} \tag{5}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{6}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \tag{7}$$

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{c}\big(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}) - \mathbf{d}\big(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \mathbf{b}\big(\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d}) - \mathbf{a}\big(\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}) \end{aligned} \tag{8}$$

Vektorski diferencijalni operator $\boldsymbol{\nabla}$ (‘nabla’) u pravokutnom koordinatnom sustavu

Definicija operatora:

$$\boldsymbol{\nabla} \equiv \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\,. \tag{9}$$

Gradijent skalarnog polja ψ (vektorska veličina):

$$\boldsymbol{\nabla}\psi = \mathbf{i}\,\frac{\partial \psi}{\partial x} + \mathbf{j}\,\frac{\partial \psi}{\partial y} + \mathbf{k}\,\frac{\partial \psi}{\partial z} \tag{10}$$

Divergencija vektorskog polja $\mathbf{A}$ (skalarna veličina):

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{11}$$

Rotacija vektorskog polja $\mathbf{A}$ (vektorska veličina):

$$\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad (12)$$

Laplace-ov operator (dif. operator drugog reda):

$$\nabla^2 \psi = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \quad (13)$$

$$\nabla^2 \mathbf{A} = \mathbf{i}\, \nabla^2 A_x + \mathbf{j}\, \nabla^2 A_y + \mathbf{k}\, \nabla^2 A_z \quad (14)$$

Diferencijalni operator ∇ u sfernim koordinatama (r, θ, ϕ)

$$\boldsymbol{\nabla}\psi = \mathbf{e}_r \frac{\partial \psi}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi} \tag{15}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\theta} \tag{16}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{A} &= \mathbf{e}_r \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] \\ &\quad + \mathbf{e}_\theta \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \mathbf{e}_\phi \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\end{aligned} \tag{17}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\,, \tag{18}$$

[Napomena: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$]

Vektorski identiteti (2)

Za vektorska polja $\mathbf{a}$ i $\mathbf{b}$, te za skalarno polje ψ vrijedi:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0 \tag{19}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) = 0 \tag{20}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) - \nabla^2\mathbf{a} \tag{21}$$

$$\boldsymbol{\nabla} \cdot (\psi\mathbf{a}) = \mathbf{a} \cdot \boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla} \cdot \mathbf{a} \tag{22}$$

$$\boldsymbol{\nabla} \times (\psi\mathbf{a}) = \boldsymbol{\nabla}\psi \times \mathbf{a} + \psi\boldsymbol{\nabla} \times \mathbf{a} \tag{23}$$

$$\boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b} + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} + \mathbf{a} \times (\boldsymbol{\nabla} \times \mathbf{b}) + \mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{a}) \tag{24}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) - \mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b}) \tag{25}$$

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla} \cdot \mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla} \cdot \mathbf{a}) + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b} \tag{26}$$

Vektorski identiteti (3)

Ako je $\mathbf{x}$ koordinata neke točke u odnosu na ishodište, $r = |x|$, $\mathbf{n} = \mathbf{x}/r$, vrijedi:

$$\boldsymbol{\nabla} \cdot \mathbf{x} = 3 \tag{27}$$

$$\boldsymbol{\nabla} \times \mathbf{x} = 0 \tag{28}$$

$$\boldsymbol{\nabla} \cdot \mathbf{n} = \frac{2}{r} \tag{29}$$

$$\boldsymbol{\nabla} \times \mathbf{n} = 0 \tag{30}$$

Za proizvoljni vektor $\mathbf{a}$ vrijedi:

$$(\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{n} = \frac{1}{r}\big(\mathbf{a} - \mathbf{n}(\mathbf{a} \cdot \mathbf{n})\big) \equiv \frac{\mathbf{a}_\perp}{r} \tag{31}$$

Teorem o gradijentu

Neka je C krivulja s krajnjim točkama P i Q, a $d\mathbf{l}$ neka je element krivulje. Za vektorsko polje koje se može izraziti kao $\mathbf{A} = \boldsymbol{\nabla}\psi$ gdje je ψ skalarno polje vrijedi:

$$\int_P^Q \mathbf{A} \cdot d\mathbf{l} = \int_P^Q \boldsymbol{\nabla}\psi \cdot d\mathbf{l} = \psi(Q) - \psi(P) \tag{32}$$

Za zatvorenu krivulju C vrijedi:

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \oint_C \boldsymbol{\nabla}\psi \cdot d\mathbf{l} = 0 \tag{33}$$

Gaussov teorem

Volumen V je omeđen zatvorenom plohom S. Element volumena je d^3x, element plohe je da, $\mathbf{n}$ je jedinični vektor okomit na da i usmjeren 'prema van'.

Gaussov teorem (teorem o divergenciji):

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \tag{34}$$

$$\int_V \boldsymbol{\nabla} \psi\, d^3x = \int_S \psi \mathbf{n}\, da \tag{35}$$

$$\int_V \boldsymbol{\nabla} \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da \tag{36}$$

Greenovi identiteti

Volumen V je omeđen zatvorenom plohom S. Element volumena je d^3x, element plohe je da, $\mathbf{n}$ je jedinični vektor okomit na da i usmjeren ‘prema van’.

Prvi Greenov identitet:

$$\int_{\mathrm{V}} (\phi\nabla^2\psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi)\, d^3x = \int_{\mathrm{S}} \phi\mathbf{n} \cdot \boldsymbol{\nabla}\psi\, da \tag{37}$$

Greenov teorem:

$$\int_{\mathrm{V}} (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_{\mathrm{S}} (\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi) \cdot \mathbf{n}\, da \tag{38}$$

Stokesov teorem

Neka je S otvorena ploha omeđena (zatvorenom) krivuljom C.
Element plohe je da, $\mathbf{n}$ je jedinični vektor okomit na da.
Element krivulje $d\mathbf{l}$ orijentiran je 'pravilom desne ruke'
u odnosu na orijentaciju vektora $\mathbf{n}$.

Stokesov teorem:

$$\int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot\, d\mathbf{l} \tag{39}$$

$$\int_S \mathbf{n} \times \boldsymbol{\nabla}\psi\, da = \oint_C \psi\, d\mathbf{l} \tag{40}$$

Klasična elektrodinamika

Vektori $\mathbf{E}$ i $\mathbf{B}$ su električno i magnetsko polje,
ρ je gustoća električnog naboja, $\mathbf{j}$ je gustoća električne struje.

Maxwellove jednadžbe:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{41}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{42}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{43}$$

$$c^2 \, \boldsymbol{\nabla} \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t} \tag{44}$$

Lorentzova sila

Na česticu naboja q koja se giba brzinom $\mathbf{v}$ u električnom polju jakosti $\mathbf{E}$ i u magnetskom polju jakosti $\mathbf{B}$ djeluje sila:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{45}$$