Laboratory exercise 2: Porpagation of measurement uncertainties

FER/Algebra

March 11, 2021

One often needs to compute the value of a quantity that depends on one or several quantites whose values have been determined by measurement. As the values of all measured quantites are subject to uncertainties, so will be the value of the dependent quantity. In this exercise, you will learn how to compute the uncertainty of a quantity that depends on measured quantities, or as we sometimes say, "how the uncertainties propagate into the dependent quantites". You will first apply the "uncertainty propagation formula" in a very simple situation, after which you will perform the measurements of two real objects and compute their volumes as well as their uncertainties.

Your report should be handed in on March 26 2021 (Friday).

1 The "uncertainty propagation formula"

If the quantity f depends on quantities $x, y, \dots, z$, we can represent f by the function

$$f = f(x, y, \dots, z). \tag{1}$$

If the values of $x, y, \dots, z$ are obtained through measurement, then for each of them we have the best estimate of its value, denoted with the barred symbols $\bar{x}, \bar{y}, \dots, \bar{z}$, as well as the estimate of its uncertainty, denoted with $\sigma_x, \sigma_y, \dots, \sigma_z$, or in full glory

$$x = \bar{x} \pm \sigma_x, \qquad y = \bar{y} \pm \sigma_y, \qquad \dots \qquad z = \bar{z} \pm \sigma_z. \tag{2}$$

The best estimate $\bar{f}$ of the value of f is then given by

$$\bar{f} = f(\bar{x}, \bar{y}, \dots, \bar{z}), \tag{3}$$

while the related uncertainty is given by

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \cdots + \left(\frac{\partial f}{\partial z}\right)^2 \sigma_z^2} \tag{4}$$

In the above expression, $\partial f / \partial x$ is the partial derivative of $f(x, y, \dots, z)$ with respect to x, etc. The expression (4) is to be evaluated using best estimates $\bar{x}, \bar{y}, \dots, \bar{z}$ of the values of $x, y, \dots, z$.

Example 1.1 *Determination of the density of a stone ball*

The results of the measurement of the mass m and the diameter d of a ball made of stone presented in the standard form read

$$m = (7.905 \pm 0.063)\,\mathrm{g}, \qquad d = 2r = (1.804 \pm 0.015)\,\mathrm{cm}. \tag{5}$$

We will show how the density ρ of the stone the ball is made of can be determined from the above data.

The density is given by

$$\rho = \frac{m}{V} = \frac{m}{\frac{4}{3}r^3\pi} = \frac{3m}{4(d/2)^3\pi} = \frac{6m}{d^3\pi}, \tag{6}$$

so the function that we will use for ρ is

$$\rho = \rho(m, d) = \frac{6m}{d^3\pi}.$$

According to (3), the best estimate $\bar{\rho}$ of the value of ρ is

$$\bar{\rho} = \rho\left(\bar{m}, \bar{d}\right) = \frac{6\bar{m}}{\bar{d}^3\pi} = \frac{6 \times (7.905\,\mathrm{g})}{(1.804\,\mathrm{cm})^3 \times \pi} = 2.5715\,\frac{\mathrm{g}}{\mathrm{cm}^3},$$

while according to (4), the estimated uncertainty σ_ρ of $\bar{\rho}$ is

$$\begin{aligned}
\sigma_\rho = \sqrt{\left(\frac{\partial\rho}{\partial m}\right)^2 \sigma_m^2 + \left(\frac{\partial\rho}{\partial d}\right)^2 \sigma_d^2} &= \sqrt{\left(\frac{6}{\bar{d}^3\pi}\right)^2 \sigma_m^2 + \left(-3\,\frac{6\bar{m}}{\bar{d}^4\pi}\right)^2 \sigma_d^2} \\
&= \frac{6\bar{m}}{\bar{d}^3\pi}\sqrt{\left(\frac{\sigma_m}{\bar{m}}\right)^2 + \left(\frac{3\sigma_d}{\bar{d}}\right)^2} \\
&= \left(2.5715\,\frac{\mathrm{g}}{\mathrm{cm}^3}\right)\sqrt{\left(\frac{0.063\,\mathrm{g}}{7.905\,\mathrm{g}}\right)^2 + \left(\frac{3 \times 0.015\,\mathrm{cm}}{1.804\,\mathrm{cm}}\right)^2} \\
&= 0.06734\,\frac{\mathrm{g}}{\mathrm{cm}^3}.
\end{aligned}$$

The density of stone can therefore be written in the standard form as

$$\rho = (2.57 \pm 0.07)\,\frac{\mathrm{g}}{\mathrm{cm}^3}.$$

The so-called relative uncertainties of the measured quantities,

$$\frac{\sigma_m}{\bar{m}} \simeq 0.8\,\%, \qquad \frac{\sigma_d}{\bar{d}} \simeq 0.8\,\%,$$

can be compared to the relative uncertainty of the dependent quantity ρ,

$$\frac{\sigma_\rho}{\bar{\rho}} \simeq 3\,\%.$$

As we can see, the relative uncertainty of the dependent quantity ρ turned out to be much larger than the relative uncertainties of the measured quantities m and d that it depends on.

Exercise 1.1: *The volume of a cone*

In the repository `http://sail.zpf.fer.hr/labs/algebra2021/L2/`, in the file `E-1-1-ddd.pdf`, where `ddd` is your personal three-digit code, you will find the values of the measurement of the base diameter d and the height h of a cone. The volume of the cone is given by the well known formula

$$V = \frac{1}{3}Sh = \frac{1}{3}r^2\pi h = \frac{1}{3}\left(\frac{d}{2}\right)^2\pi h = \frac{\pi}{12}\,d^2 h,$$

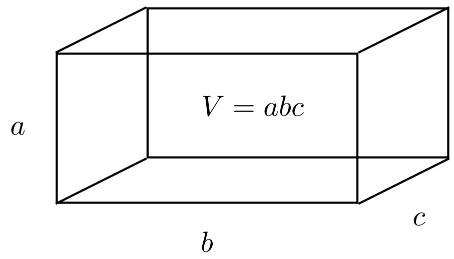
where $S = r^2\pi$ is the base area, and $r = d/2$ is the base radius. In this exercise you should:

- Compute the best estimate $\bar{V}$ of the volume of the cone.
- Derive the formula for the uncertainty σ_V of $\bar{V}$ and evaluate it.
- Present V in the standard form using one significant digit for the uncertainty.
- Compute the relative uncertainties of d and h.
- Compute the relative uncertainty of V and compare it to the relative uncertainties of d and h.

2 Determination of the volume of two physical objects

Exercise 2.1: *The volume of a body with the shape of a rectangular coboid*

You should chose an object that has the shape of a rectangular cuboid (its 6 faces are rectangles), measure the length of its three sides, and determine its volume.



The object could be

- the inside of the room in which you are working or some nearby room,
- the outside or the inside of a cardboard box (e.g. a shoe box),
- a paperback book, e.g. your physics textbook (hopefully you'll receive the hardcopy, pdf is no good for this), etc.

Of course, you should also find an adequate measurement device. If you are dealing with a small object, a ruler could do the job, while if you are measuring the room, a tape meter would be much better. In order to determine the volume of your object, you should perform the following:

- Measure the length of each of the three sides at least ten times, preferably at different places.
- For each side determine the best estimate of its length and the related uncertainty.
- Present each length in the standard form rounding the uncertainty to one significant digit.
- Determine the relative uncertainty for the length of each side (see Example 1.1).
- Determine the best estimate of the volume of the object (formula 3).
- Use the "uncertainty propagation formula" (formula 4) to determine the uncertainty of the value of the volume computed above.
- Determine the relative uncertainty of the volume and compare it to the relative uncertainties you obtained for the three sides.

If you have any comments on the procedure you have been through, or you have any points you would like to discuss, please note them in your report so that you do not forget about them. We'll be happy to read.

Exercise 2.2: *The volume of a size–AA battery or similar*

The shape of a size-AA battery or similar (e.g. D, C, AAA) can be aproximatd as two cylinders:

- the large cylinder (the main body of the battery), and
- a small cylinder attached to one end of the large cylinder (the positive terminal of the battery).

For each of the two cylinders you should:

- Measure the diameter d and the length ℓ of the cylinder, and present the results in the standard form (use one significant digit for the uncertainty).
- Determine the volume and its uncertainty, and present the result in the standard form (use one significant digit for the uncertainty).

Note: While measuring the diameter and the length of the small cylinder you might run into difficulties due to small size of the cylinder and insufficient resolution of your measurement equipment. For example, all 10 repeated measurements could give the same value. If this happens, you are allowed to estimate the uncertainty using your intuition. For example, if $1\,\mathrm{mm}$ is the resolution of your ruler, as the uncertainty of the measurement you can take $\sigma = 0.5\,\mathrm{mm}$

You should now attempt to determine the volume of the battery by adding up the volumes of the two cylinders. Here you can consider the volumes of the two cylinders as the two measured quantities, while their sum can be seen as the dependent quantity. Present the battery volume in the standard form with one significant digit in the uncertainty. Try to address the following question:

- Is the sum of the two volumes, when the result is presented in the standard form (use one significant digit for the uncertainty), any different from the volume of the large cylinder alone?

If not, how can that be? Why is that so? Please discuss. Again, we are interested to hear about your observations.