Laboratory exercise 1: Measurememts in physics

FER/Algebra

February 28, 2021

In this exercise, you will learn about uncertainties that are an unavoidable feature of all measurements in physics and engineering. You will gain practice with some basic computational procedures that lead to the best estimates of measured values, as well as to their uncertainties. Most of your work will be of computational nature, but you will also have to carry out some real measurements at your home. The last section on the method of least squares is intended only for curious students.

The report should be handed in on March 12 2021 (Friday).

1 The uncertainty of measurement

Probably the most important property of any measurement in physics and engineering is that the value of a measured quantity can never be determined with absolute certainty. The result of a typical measurement consists of two things:

- the best estimate of the value of the quantity being measured,
- and the estimate of the uncertainty of the measurement.

For example, a recently reported value of the Hubble constant (a quantity that has to do with the rate of the expansion of the Universe) reads

$$H = (73.2 \pm 1.3)\,\frac{\mathrm{km/s}}{\mathrm{kpc}}.$$

Here $73.2\ (\mathrm{km/s})/\mathrm{kpc}$ is the best estimate of H the researchers could obtain through the methods that they used, while $\pm 1.3\ (\mathrm{km/s})/\mathrm{kpc}$ is the estimate of the uncertainty of their measurement. Another example is the mass of the famous Higgs boson (an elementary particle found at CERN in 2012 after decades of experimental search) which reads

$$m_H = (125.10 \pm 0.14)\ \mathrm{GeV}/c^2.$$

In both cases, regardless of all efforts that have been made by the researchers to measure these quantities as accurately and as precisely as possible, the measurement uncertainty remains relatively large and is always reported. When presenting a measured quantity the following rule is usually obeyed:

- The measurement uncertainty is rounded up so that it contains only one or at most two significant digits.[1]
- The estimated value of the measured quantity is rounded up to the exactly same decimal place as the uncertainty has been rounded up.

[1] In case you are not familiar with the concept of significant digits in the decimal representation of a number you may refer to Chapter 1, Section 5, of your textbook (Young & Freedman 2019).

If the above rule is followed, we say that the result of a measurement is presented in the *standard form*.

Example 1.1: *Presenting a measurement in the standard form*

Repeated measurements of the mass of a steel ball gave

$$\bar{m} = 4.08607 \text{ g}$$

as the best estimate of the ball mass, while the uncertainty of this measurement was estimated to

$$\sigma_m = 0.02732 \text{ g}.$$

The result of this measurement presented in the standard form using two significant digits for the uncertainty would read

$$m = (4.086 \pm 0.027) \text{ g},$$

while with only one significant digit in the uncertainty it would read

$$m = (4.09 \pm 0.03) \text{ g}.$$

Note that the estimated value is always rounded at the same decimal place as the uncertainty is rounded.

Exercise 1.1: *Presenting measurements in the standard form*

In the repository `http://sail.zpf.fer.hr/labs/algebra2021/L1/`, in the file `E-1-1-ddd.pdf`, where `ddd` is your personal three-digit code that you were assigned at the beginning of the course, you will find the best estimates and the uncertainties obtained in five unrelated experiments. This file should be included in your report.

In this exercise you should:

- present each of the given results in the standard form (see Example 1.1) first rounding up the uncertainty to two,
- and then to only significant digit.

You should pay special attention to proper rounding up of values and do not forget to include the appropriate measurement unit.

2 Independent measurements of a quantity

The simplest measurement scenario is the one in which you rightfully assume that the quantity you are measuring is constant, and you are in the position to repeat the measurement as many times as you like. Such measurements are said to be independent measurements. Typically, independent measurement will differ among themselves, the variations being due to the intrinsic variations of the quantity being measured, due to imperfection of your equipment, or due to noise, mechanical vibrations, electrical interference, or any other influence from the environment. Let $y_1, y_2, \ldots, y_N$, or as we usually write

$$y_i, \qquad i = 1, \ldots, N, \tag{1}$$

be the N independent measurements of a quantity y. We will refer to (1) simply as the *data*. As you may intuitively expect, the best estimate of the measured quantity value in this case is the *arithmetic mean* of the data,

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i = \frac{y_1 + y_2 + \cdots + y_N}{N}. \tag{2}$$

According to the statistical theory that we shall not go into, the uncertainty of this measurement is given by the quantity that is known as the *standard deviation of the mean* and is given by the formula

$$\sigma_{\bar{y}} = \sqrt{\frac{1}{N(N-1)}\sum_{i=1}^{N}(y_i - \bar{y})^2} \tag{3}$$

The conclusion of your measurements is summarized and specified in the standard form as

$$y = \bar{y} + \sigma_{\bar{y}}, \tag{4}$$

taking proper care to round the numerical values as explained in the preceding section and not forgetting to specify the appropriate measurement unit. The process of downsizing a large data set like (1) into a compact and comprehendible conclusion like (4) is called *data reduction*.

Example 2.1: *Reduction of independent measurements*

A person stepped onto the body weight scale ten times and took note of individual scale readings. The $N = 10$ independent measurements m_i of his/her mass m in kilograms are:

$$74.3, \quad 74.0, \quad 73.8, \quad 74.2, \quad 74.1, \quad 74.1, \quad 73.7, \quad 73.8, \quad 74.1, \quad 74.3.$$

Rather disappointed and confused with more than obvious variation among the scale readings, the person decided to reduce the data and present the result in the standard form. The best estimate of measured body mass is given by the mean of the independent measurements,

$$\bar{m} = \frac{1}{N}\sum_{i=1}^{N} m_i = \frac{740.4\ \mathrm{kg}}{10} = 74.04\ \mathrm{kg},$$

while the uncertainty of the measurement is given by the standard deviation of the mean,

$$\sigma_{\bar{m}} = \sqrt{\frac{1}{N(N-1)}\sum_{i=1}^{N}(m_i - \bar{m})^2} = \sqrt{\frac{0.404\ \mathrm{kg}^2}{10 \times (10-1)}} = 0.0669992\ \mathrm{kg}.$$

The final result written in the standard form with only one significant digit in the uncertainty is

$$m = (74.04 \pm 0.07)\ \mathrm{kg}.$$

Exercise 2.1: *Reduction of independent measurements*

In the repository `http://sail.zpf.fer.hr/labs/algebra2021/L1/`, in the file `E-2-1-ddd.pdf`, where `ddd` is your personal three-digit code, you will find a number of values of a physical quantity that have been obtained in independent measurements, as well as the related measurement unit. This file should be included in your report.

In this exercise you should:

- compute the best estimate of the measured quantity value (the mean),

- compute the uncertainty of the measurement (standard deviation of the mean), and
- present the result in the standard form with one significant digit in the uncertainty.

Exercise 2.2: *A do-it-yourself experiment involving independent measurements*

You should think of a simple measurement that you can make yourself using any equipment that is available to you and which you can repeat at least 10 times or more. You could consider measuring one of the following:

- your or somebody else's body mass using standard body weight scale (see Example 2.1),
- mass of some small object using the kitchen scale (typically range up to $4\ \mathrm{kg}$),
- your body temperature using a standard medical thermometer,
- circumference of a long object with a regular cross section (leg of a table, some pipe, a beer can) by wrapping a sheet of paper around it, putting marks with a pencil, unwrapping the sheet, flattening it out, and measuring the distance between the marks,
- upload or download speed of your Internet link using some of the publicly available software tools for whatever device you are using,

or come up with something more interesting of your own. You should describe your experimental setup and provide a photograph if possible. You should obtain at least 10 independent measurements, reduce them and present the final result in the standard form.

3 Measurements of a quantity depending on another quantity

In many experiments in physics and engineering, one is measuring a physical quantity that depends on one or more other physical quantities that change in time and can also be measured, or that can be set to the desired values. In this way the dependence of a physical quantity on other quantities can be investigated.

For simplicity, here we will consider only the situation in which it is reasonable to expect that a quantity y depends linearly on a quantity x. Such dependence can be expressed mathematically as

$$y(x) = ax + b, \tag{5}$$

where a and b are the unknown coefficients whose values are to be determined from the measurements. We usually say that equation (5) defines the *model* for the dependence of y on x, and that a and b are the *model parameters*. Let us assume that we measured N pairs of values of x and y, which means that our data has the form

$$(x_i, y_i), \qquad i = 1, \ldots, N. \tag{6}$$

According to the statistical theory on which we shall not elaborate, using (5) as the model for our data, the best estimates of values of the model parameters a and b are given by

$$a = \frac{1}{\Delta}\left(N\sum_i x_i y_i - \sum_i x_i \sum_i y_i\right), \qquad b = \frac{1}{\Delta}\left(\sum_i x_i^2 \sum_i y_i - \sum_i x_i \sum_i x_i y_i\right), \tag{7}$$

where

$$\Delta = N\sum_i x_i^2 - \left(\sum_i x_i\right)^2. \tag{8}$$

The uncertainties of a and b are given by

$$\sigma_a = \sigma_y \sqrt{\frac{N}{\Delta}}, \qquad \sigma_b = \sigma_y \sqrt{\frac{1}{\Delta} \sum_i x_i^2}, \tag{9}$$

where Δ is given by (8) and

$$\sigma_y = \sqrt{\frac{1}{N-2} \sum_i \left(y_i - (a x_i + b)\right)^2}. \tag{10}$$

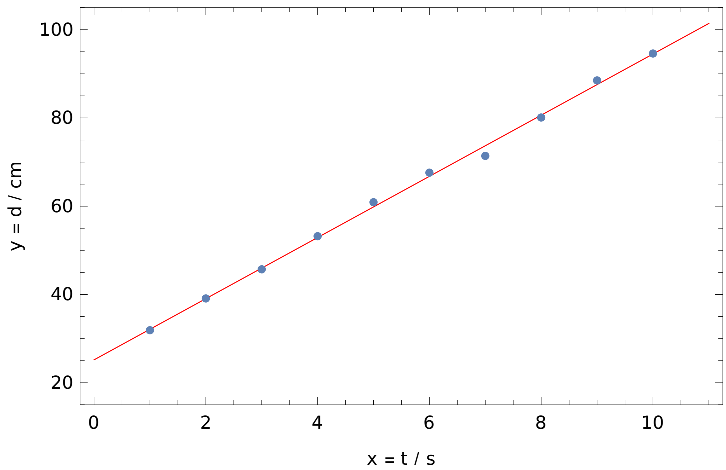
Once the values of the parameters a and b and their uncertainties have been determined for some data set, we usually say that we performed a measurement of a and b.

Example 3.1: *Determination of the speed of an object*

An object is moving along a straight horizontal rail and its distance d from one end of the rail is being measured by a sonar in time instants separated by one second. The measurement data is given in the following table:

t_i/s	1	2	3	4	5	6	7	8	9	10
d_i/cm	31.9	39.1	45.7	53.2	60.9	67.6	71.4	80.1	88.5	94.6

The data can be shown graphically using the coordinates $x = t/\mathrm{s}$ (time in seconds) and $y = d/\mathrm{cm}$ (position in centimetres). Blue bullets are the measurements.



It is easy to see that our $N = 10$ data points roughly sit on a line. Therefore it is reasonable to assume that the body is moving at a constant speed, and that the small deviations of the data points may be due to the imperfection of the measurement procedure (sonar). We can therefore assume the model

$$y_i = a x_i + b,$$

where the parameter a is the speed in centimetres per second, and the parameter b can be interpreted as the position where the body could have been at $t = 0$. In order to determine the best estimates of the values of the parameters a and b we first compute the sums

$$\sum_i x_i = 55, \qquad \sum_i y_i = 633, \qquad \sum_i x_i y_i = 4053.2, \qquad \sum_i x_i^2 = 385,$$

then we use (8) to compute

$$\Delta = 825,$$

and finally we use (7) to obtain

$$a = 6.9297 \qquad \text{and} \qquad b = 25.1867.$$

The red line in the above graph was drawn using the above values of a and b.

Proceeding to the uncertainties of a and b, we first use (10) to compute

$$\sigma_y = 1.03031,$$

which is needed in (9) to obtain

$$\sigma_a = 0.113433 \qquad \text{and} \qquad \sigma_b = 0.703833.$$

Finally the speed of the body as determined from the measurements and the model is

$$v = (a \pm \sigma_a)\ \mathrm{cm/s} = (6.9 \pm 0.1)\ \mathrm{cm/s}.$$

We may refer to this final result as the measured speed of the object.

Exercise 3.1: *Determination of the speed of an object*

In the repository `http://sail.zpf.fer.hr/labs/algebra2021/L1/`, in the file `E-3-1-ddd.pdf`, where `ddd` is your personal three-digit code, you will find a data set similar to the one that was considered in Example 3.1. This file should be included in your report.

You should carry out yourself the procedure of determining the speed of the object using the data from the file. In your report you should:

- plot the data,
- compute and document all the intermediate results (values of various sums, Δ, σ_y) that are needed to compute a, b, and the uncertainties,
- explain shortly how you computed the values (did you use a pocket calculator, a spreadsheet program on a computer, or some programming language, if so which?),
- provide unrounded values of a, b, σ_a and σ_b,
- present the final result for the speed of the body in the standard form using one significant digit for the uncertainty, and
- provide a plot that contains both data and the model (line).

The plots can be drawn by hand or by a computer program of your choice.

4 Least squares method

You certainly noticed that the expressions (7) for best estimates of model parameters values and especially expressions (9) for the uncertainties were presented to you as a “recipe”, i.e. without any theoretical justification. The reason for this is that the statistical theory needed to derive these expressions requires a thorough discussion of an intricate set of subtle assumptions that is outside of the scope of the present physics course. However, there is one remarkably elegant concept known as *the least squares method* which allows you to obtain the best estimates of model parameter values in the cases that we considered, as well as in many other situations.

Let us assume that the dataset consists of N measurements of quantities x and y that depend one on another,

$$(x_i, y_i), \qquad i = 1, \dots, N, \tag{11}$$

and that the model for the data can be expressed as

$$y(x; p, \dots, q), \tag{12}$$

where $p, \dots, q$ are the unknown model parameters.[2] We seek to find the values of model parameters that make the model fit the data as closely as possible. To this end we first introduce the quantity δ_i known as the *residual* of the data point i as the deviation of y_i from the value predicted by the model,

$$\delta_i = y_i - y(x_i; p, \dots, q), \qquad i = 1, \dots, N. \tag{13}$$

and proceed to construct a quantity that gives us a measure of how well the model fits the whole dataset. This quantity is constructed as the sum of squared residuals

$$S^2(p, \dots, q) = \sum_{i=1}^{N} \delta_i^2 = \sum_{i=1}^{N} \big(y_i - y(x_i, p, \dots, q)\big)^2, \tag{14}$$

and is known as *sum-of-squares* for short. The least squares method is based on the assumption that the model that fits the data the best is the one that makes the sum-of-squares the least. Technically, this means that we must minimize S^2 relative to the model parameters, which implies one equation for each parameter,

$$\frac{\partial}{\partial p} S^2(p, \dots, q) = 0, \qquad \dots \qquad \frac{\partial}{\partial q} S^2(p, \dots, q) = 0. \tag{15}$$

Solving the above system of coupled equations for the model parameters $p, \dots, q$ gives us the expressions for best estimates of their values.

Example 4.1: *Independent measurements via least squares*

In the case of independent measurements of y we are not considering the dependence y on any other quantity so the dataset is simply y_i, $i = 1, \dots, N$, and the data model is $y = c$. The residuals are therefore $\delta_i = y_i - c$, and the sum-of-squares is

$$S^2 = \sum_i \delta_i^2 = \sum_i (y_i - c)^2 = \sum_i \left(y_i^2 - 2cy_i + c^2\right) = \sum_i y_i^2 - 2c \sum_i y_i + Nc^2.$$

The last expression is quadratic in c with a minimum at

$$c = \frac{1}{N} \sum_i y_i,$$

which we recognize as the arithmetic mean of the data. We have shown that according to the least squares method the mean of the independent measurements is the best estimate of the value of measured quantity, exactly as we stated in Section 2, equation (2).

[2] We also tacitly make the following assumptions: (i) the uncertainties of the individual measurements of y relative to the range covered by all measurements of y are considerably larger than the uncertainties of x relative to the range covered by all measurements of x, and (ii) we assume that uncertainties of individual measurements of y are approximately uniform.

Example 4.2: *Linear dependence of y on x via least squares*

Assuming y is linear in x, the data model can be written as $y = ax + b$, and the sum-of-squares can be written out as

$$S^2 = \sum_i \delta_i^2 = \sum_i \big(y_i - (ax_i + b)\big)^2 =$$

$$\sum_i y_i^2 - 2a \sum_i x_i y_i + a^2 \sum_i x_i^2 + 2ab \sum_i x_i + Nb^2 - 2b \sum_i y_i.$$

Following the least squares method, we require that the partial derivatives of S^2 with respect to parameters a and b vanish, which gives us

$$0 = \frac{\partial}{\partial a} S^2 = -2 \sum_i x_i y_i + 2a \sum_i x_i^2 + 2b \sum_i x_i,$$

and

$$0 = \frac{\partial}{\partial b} S^2 = 2a \sum_i x_i + 2Nb - 2 \sum_i y_i.$$

If these two equations are solved for a and b, equations (7) of Section 3 are obtained as solutions.

In data reduction in physics one often needs more complex data models than those considered here. For example, a very useful data model is the quadratic,

$$y(x) = \frac{a}{2} x^2 + bx + c,$$

which is linear in the parameters a, b, and c, and the expressions for best estimates of values of these parameters can be obtained. However, these expressions are quite lengthy and will not be given here.

Data models that are nonlinear in their parameters are more difficult to handle because the solutions are not accessible in closed form, numerical optimization procedures must be used, and the solutions are not guaranteed to be unique. However, there are some important special cases in which simple mathematical transformations can convert a nonlinear problem into a linear one for which the least squares method gives the result straightforwardly. We conclude this section with one such example.

Example 4.3: *Exponential fall-off of y with x*

If a laser beam is passing through some homogeneous absorbing medium, then the intensity of the beam falls off (becomes weaker) according to the law

$$I(x) = I_0 \exp(-\beta x),$$

where paraeter I_0 is the initial beam intensity, the parameter β describes the strength of the absorption (larger β means more absorption per unit length), and x is the length of the path that the beam has travelled through the medium. Let us assume that we measured (I_i, x_i), $i = 1, \ldots, N$, and that we would like to determine the value of β. Writing out the sum-of-squares using I_i and x_i gives us

$$S^2 = \sum_i \left(I_i - I_0 \exp(-\beta x)\right)^2,$$

which would not, if we followed the least squares method, take us to a system of linear equations in I_0 and β that we can solve. If, on the other hand, we define a new variable $y = \ln I$, then the model becomes linear and reads

$$y = y_0 - \beta x,$$

where $y_0 = \ln I_0$ is the new parameter, and the new sum-of-squares using y instead of I is

$$S'^2 = \sum_i \left(y_i - (y_0 - \beta x_i)\right)^2 .$$

It is easy to see that this is exactly the problem we considered in Example 4.2, the only difference being in the notation.